Latin Squares: Critical Sets

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Author: Tamara Gomez **Definition.** A **Latin square** is an $n \times\ n$ grid filled with n distinct symbols (usually $\{1, \ldots, n\}$) so that no symbol is repeated in any row or column.

Example. Here is a Latin square of order 3:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 3 \end{bmatrix}$$

An interesting topic of discussing involving Latin squares are critical sets.

Definition. A **critical set** is a partial Latin square with the following two properties:

- It is completable to a unique Latin square
- If one entry is deleted, the partial Latin square is no longer uniquely completable

Example. Look at the partial Latin square:

$$\begin{bmatrix} 1 & 2 & \\ 2 & & \\ & & \end{bmatrix}$$

This partial Latin square is uniquely completable to the following Latin square

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 3 \end{bmatrix}$$

Notice that if we remove one entry from our partial Latin square, we will no longer have unique completion.
i.e the partial Latin square

$$\begin{bmatrix} 1 & & \\ 2 & & \\ & & \end{bmatrix}$$

can be completed into the Latin squares

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Therefore our partial Latin square is a critical set.

The term *critical sets* was coined by J. A. Nelder, and he asked the following question: Given a Latin square of order n, what is the size[1] of the smallest critical set (denoted by scs(n)) and the largest critical set (denoted by lcs(n))?

There are a few things known about critical sets and their sizes.

1. Critical sets of different sizes exist for the same Latin square.
   Example.
   The critical sets

$$\begin{bmatrix} 1 & 2 & & \\ 2 & & & \\ & & & \\ & & & 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 3 & \\ 2 & 3 & & \\ 3 & & & \\ & & & \end{bmatrix}$$

   both uniquely complete to the Latin square

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

   However the size of the first critical set is 4, while the size of the second critical set is 6.

2. Different Latin squares of the same order n have smallest critical sets of different sizes.
   Example. The smallest critical set

$$\begin{bmatrix} 1 & 2 & & \\ 2 & & & \\ & & & \\ & & & 3 \end{bmatrix}$$

   uniquely completes to the Latin square

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

   While the smallest critical set

$$\begin{bmatrix} & 2 & & \\ 2 & & & 3 \\ & & 1 & \\ 4 & & & \end{bmatrix}$$

[1]The size of a critical set is determined by the number of entries in the partial Latin square.

uniquely completes to the Latin square

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

Both Latin squares are of order 4, but the size of the critical set of the first Latin square is 4, while the size of the critical set of the second Latin square is 5.

But how do we know that those critical sets are indeed the smallest ones? Well funny you should ask! Look at the Latin square

$$\begin{bmatrix} \color{red}{1} & 2 & \color{red}{3} & 4 \\ \color{blue}{2} & \color{green}{3} & \color{blue}{4} & \color{green}{1} \\ \color{red}{3} & 4 & \color{red}{1} & 2 \\ \color{blue}{4} & \color{green}{1} & \color{blue}{2} & \color{green}{3} \end{bmatrix}$$

Notice that each element in this Latin square belongs to a 2×2 Latin subsquare, called an *intercalate* (each different intercalate is colored a different color). If a critical set does not include at least one cell from each intercalate, it will be possible to obtain two separate completions of that set, i.e. it would not be a critical set because it would not be uniquely completable. Therefore a critical set must include at least one cell from each intercalate. There are four intercalates in our Latin square, therefore there must be at least four cells in its smallest critical set. So the critical set

$$\begin{bmatrix} 1 & 2 & & \\ 2 & & & \\ & & & \\ & & & 3 \end{bmatrix}$$

contains a cell from each intercalate and is indeed the smallest critical set for our Latin square.

However, the second Latin square from our example

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

is a bit more complicated. Here, each cell is part of three distinct intercalates. Since there are 16 elements in our Latin square, each cell is in three distinct intercalates, and each intercalate has four cells, there are $\frac{16 \cdot 3}{4} = 12$ distinct intercalates total.

$$\begin{bmatrix} \color{red}{1} & \color{red}{2} & 3 & 4 \\ \color{red}{2} & \color{red}{1} & 4 & 3 \\ \color{blue}{3} & \color{blue}{4} & \color{green}{1} & \color{green}{2} \\ \color{blue}{4} & \color{blue}{3} & \color{green}{2} & \color{green}{1} \end{bmatrix} \quad \begin{bmatrix} \color{red}{1} & 2 & \color{red}{3} & 4 \\ \color{blue}{2} & \color{green}{1} & \color{blue}{4} & \color{green}{3} \\ \color{red}{3} & 4 & \color{red}{1} & 2 \\ \color{blue}{4} & \color{green}{3} & \color{blue}{2} & \color{green}{1} \end{bmatrix} \quad \begin{bmatrix} \color{red}{1} & \color{blue}{2} & \color{blue}{3} & \color{red}{4} \\ 2 & \color{green}{1} & \color{green}{4} & 3 \\ 3 & \color{green}{4} & \color{green}{1} & 2 \\ \color{red}{4} & \color{blue}{3} & \color{blue}{2} & \color{red}{1} \end{bmatrix}$$

Above you can see all twelve intercalates of our Latin square. Once again, in order to obtain the smallest critical set for our Latin square, the critical set must contain a cell of each of the twelve intercalates. We can do this through a *transversal* of our Latin square.
Definition. A **transversal** of a Latin square of order n is a partial Latin square with the following properties:

- n cells are filled
- each cell contains a different element
- each row and each column contains one of the elements.

A transversal of the Latin square we are looking at would be:

$$\begin{bmatrix} & 2 & & \\ & & & 3 \\ & & 1 & \\ 4 & & & \end{bmatrix}$$

because n cells are filled, each cell contains a different element, and each row and column contain one of the elements. However, notice that this partial Latin square is not a critical set because it can be completed to the following two Latin squares:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 2 & 4 & 1 \\ 1 & 4 & 2 & 3 \\ 2 & 3 & 1 & 4 \\ 3 & 1 & 3 & 2 \end{bmatrix}$$

But by filling in one more cell, we fix that problem, so

$$\begin{bmatrix} & 2 & & \\ 2 & & & 3 \\ & & 1 & \\ 4 & & & \end{bmatrix}$$

is indeed the smallest critical set for our Latin square.

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We say that an entry in a cell of a Latin square is *forced* if there is only one entry that can possibly go in that cell so that the rules of a Latin square are not broken. So if we have some n x n partial Latin square, its entries being the digits 1 through n, with some empty cell C, and the row and column containing C together contain $n-1$ different digits, we know that we can fill C with the remaining digit. For example, in the partial Latin square below, we know that the entries in the first row, first column, and second row, third column must be 3, because the two remaining digits, 1 and 2, are present in their rows and columns. So if we were to put 1 or 2 in either of those cells, our partial Latin square would no longer be a partial Latin square. So those entries are forced.

		2
1		

3		2
1		3

After we fill in those entries, other entries become forced as well: The cell in the first row, second column must contain a 1, the cell in the second row, second column must contain a 2, and so on. We can continue filling out the Latin square in this manner:

3	1	2
1	2	3
2	3	1

If a critical set, such as the one above, has the property that every entry is at some point forced while filling out the Latin square, we say that the critical set is *strongly completable*. But what if this is not the case? Consider the following critical set:

				4
	5	3		
2				
3		1		
			1	

At this point, none of the entries in this partial Latin square are forced. For example, the entry in the first row, first column could be either 1 or 5. If, when completing a critical set into a Latin square, there is some point where none of the entries are forced, we say that the critical set is *weakly completable*. If, as in the critical set above, no entries are forced at the beginning (when nothing is filled in except the critical set), we say that the critical set is *totally weak*.

So how do we complete this Latin square? We have to start by making a guess. As stated earlier, the top left cell could contain either a 1 or a 5. We could try filling in a 1, and that will force some entries, and then we will have to make another guess, but eventually we will get to a contradiction – where there are no possible options for an entry because the digits 1-5 are all in its row and column already. If we fill in a 5, however, all of the rest of the entries are forced, and we end up with the following Latin square:

5	1	2	3	4
1	5	3	4	2
2	3	4	5	1
3	4	1	2	5
4	2	5	1	3

.

Another definition: Two latin squares are *isotopic* if you can change one into the other by some combination of the following three actions: rearranging the rows, rearranging the columns, or renaming the symbols (for example, changing all the 1's to 2's and all the 2's to 1's). The three Latin squares below are isotopic. We change the first one to the second one by switching the second and third rows, and we change the second one to the third one by switching the 1's and 3's.

1	2	3
2	3	1
3	1	2

$\rightarrow$

1	2	3
3	1	2
2	3	1

$\rightarrow$

3	2	1
1	3	2
2	1	3

Here are some open problems in critical sets: Statistician John Nelder conjectured that the smallest critical set for a Latin square that was a group table for $\langle \mathbb{Z}/n\mathbb{Z}, + \rangle$ would consist of two triangles in the top left and bottom right of the square. For odd n, these triangles would both have $(n^2 - 1)/8$ entries, and for even n, one triangle would have $(n^2 + 2n)/8$ entries and the other would have $(n^2 - 2n)/8$. Here is an example when $n = 4$:

0	1		
1			
			2

This conjecture has been proven for strongly completable critical sets, but not for weakly completable ones. The question of the size of the largest critical sets for general n is also open.

Sources: "Critical Sets in Latin Squares: An Intriguing Problem", by A. Donald Keedwell, *The Mathematical Gazette*

"Minimal Critical Sets" by Landon Settle